

Continuous structures of quantum circuits

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Abstract. We consider continuous structures which are obtained from finite dimensional Hilbert spaces over \mathbb{C} by adding some unitary operators. Quantum automata and circuits are naturally interpretable in such structures. We consider appropriate algorithmic problems concerning continuous theories of natural classes of these structures.

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0 Introduction

Continuous logic has become the basic model theoretic tool for Hilbert spaces and \mathbb{C}^* -algebras: see [1], [3] and [10]. This suggests that quantum circuits, quantum automata and quantum computations in general can be defined in appropriate continuous structures and studied by means of continuous logic. The paper is an attempt of this approach.

It is worth noting that typical continuous structures appearing in quantum informatics are finite dimensional, i.e. compact and not very interesting from the point of view of model theory. On the other hand we demonstrate in our paper that study of continuous theories of *classes* of these structures may be promising.

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We remind the reader that states of quantum systems are represented by normed vectors of tensor products

$$(\dots(\mathcal{B}_1 \otimes \mathcal{B}_2) \otimes \dots) \otimes \mathcal{B}_k,$$

where $\mathcal{B}_i \cong \mathbb{C} \bigoplus \mathbb{C}$ under isomorphisms of Hilbert spaces, $i \leq k$.

In Dirac's notation elements of \mathcal{B}_i are denoted by $|h\rangle$ and tensors

$$(\dots(|h_1\rangle \otimes |h_2\rangle)\dots) \otimes |h_k\rangle \text{ are denoted by } |h_1 h_2 \dots h_k\rangle.$$

Any normed $h \in \mathcal{B}_i$ is called a **qubit**; it is a linear combination of $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$.

The probability amplitude $a(\phi \rightarrow \psi)$ is defined as the inner product $\langle \psi | \phi \rangle$ and the probability $p(\phi \rightarrow \psi)$ is $|a(\phi \rightarrow \psi)|^2$. Dynamical evolutions of the quantum system are represented by unitary operators on $\mathcal{B}^{\otimes k}$.

We call the structure $\mathcal{B}^{\otimes k}$ enriched by unitary operators U_1, \dots, U_t a **dynamical n -qubit space**. It in particular defines a family of quantum automata over the language $\{1, \dots, t\}^*$, where each automaton is determined by the 2^n -dimensional diagonal matrix P of the projection to final states. Fixing $\lambda \in \mathbb{Q}$ it is said that a word $w = i_1 \dots i_k$ is *accepted* if

$$\| P U_{i_k} \dots U_{i_1} |0^{\otimes n}\rangle \|^2 > \lambda.$$

These issues are described in [12], [13] and [5].

We will consider dynamical n -qubit spaces in continuous logic. All necessary information on continuous logic will be described in the next section. The main results of the paper (see Section 4) concern decidability of continuous theories of classes of dynamical qubit spaces. They are in particular motivated by [5] (see Section 2 below). Section 3 contains some general observations concerning decidability. We think that this section is interesting by itself. It is naturally connected with the material of [2], [6] and [11].

It is worth noting that continuous logic can be considered as a theory in some extension (**RPL** \forall) of Łukasiewicz logic (see [6]). The latter is traditionally linked with quantum mechanics, [4], [14]. Thus the idea that continuous logic should enter into the field is quite natural. On the other hand the author thinks that the context of the paper is original.

1 Continuous structures.

We fix a countable continuous signature

$$L = \{d, R_1, \dots, R_k, \dots, F_1, \dots, F_l, \dots\}.$$

Let us recall that a *metric L -structure* is a complete metric space (M, d) with d bounded by 1, along with a family of uniformly continuous operations on M and

a family of predicates R_i , i.e. uniformly continuous maps from appropriate M^{k_i} to $[0, 1]$. It is usually assumed that to a predicate symbol R_i a continuity modulus γ_i is assigned so that when $d(x_j, x'_j) < \gamma_i(\varepsilon)$ with $1 \leq j \leq k_i$ the corresponding predicate of M satisfies

$$|R_i(x_1, \dots, x_j, \dots, x_{k_i}) - R_i(x_1, \dots, x'_j, \dots, x_{k_i})| < \varepsilon.$$

It happens very often that γ_i coincides with id . In this case we do not mention the appropriate modulus. We also fix continuity moduli for functional symbols. Note that each countable structure can be considered as a complete metric structure with the discrete $\{0, 1\}$ -metric.

By completeness continuous substructures of a continuous structure are always closed subsets.

Atomic formulas are the expressions of the form $R_i(t_1, \dots, t_r)$, $d(t_1, t_2)$, where t_i are terms (built from functional L -symbols). In metric structures they can take any value from $[0, 1]$. *Statements* concerning metric structures are usually formulated in the form

$$\phi = 0$$

(called an *L-condition*), where ϕ is a *formula*, i.e. an expression built from 0,1 and atomic formulas by applications of the following functions:

$$\begin{aligned} x/2, x \dot{-} y &= \max(x - y, 0), \min(x, y), \max(x, y), |x - y|, \\ \neg(x) &= 1 - x, x \dot{+} y = \min(x + y, 1), x \cdot y, \sup_x \text{ and } \inf_x. \end{aligned}$$

A *theory* is a set of *L-conditions* without free variables (here \sup_x and \inf_x play the role of quantifiers).

It is worth noting that any formula is a γ -uniformly continuous function from the appropriate power of M to $[0, 1]$, where γ is the minimum of continuity moduli of *L*-symbols appearing in the formula.

The condition that the metric is bounded by 1 is not necessary. It is often assumed that d is bounded by some rational number d_0 . In this case the (dotted) functions above are appropriately modified. Sometimes predicates of continuous structures map M^n to some $[q_1, q_2]$ where $q_1, q_2 \in \mathbb{Q}$.

Remark 1.1 Following Section 4.2 of [10] we define a topology on *L*-formulas relative to a given continuous theory T . For n -ary formulas ϕ and ψ of the same sort set

$$\mathbf{d}_{\bar{x}}^T(\phi, \psi) = \sup\{|\phi(\bar{a}) - \psi(\bar{a})| : \bar{a} \in M, M \models T\}.$$

The function $\mathbf{d}_{\bar{x}}^T$ is a pseudometric. The language L is called *separable* if for every *L*-theory T and any tuple \bar{x} the density character of $\mathbf{d}_{\bar{x}}^T$ is countable. By Proposition 4.5 of [10] in this case for every *L*-model M the set of all interpretations of *L*-formulas in M is separable in the uniform topology.

The paper [2] gives fourteen axioms of continuous first order logic, denoted by (A1) - (A14), and the corresponding version of *modus ponens*:

$$\frac{\phi, \psi \dot{-} \phi}{\psi}. \text{ where } \phi, \psi \text{ are continuous formulas.}$$

Corollary 9.6 of [2] states

Let Γ be a set of continuous formulas of a continuous signature L with a metric. Let ϕ be a continuous L -formula. Then the following conditions are equivalent:

- (i) for any continuous structure M and any M -assignment of variables, if M satisfies all statements $\psi = 0$, $\psi \in \Gamma$, then M satisfies $\phi = 0$;
- (ii) $\Gamma \vdash \phi \dot{-} 2^{-n}$ for all $n \in \omega$.

It is called **approximated strong completeness for continuous first-order logic**. The following statement is Corollary 9.8 from [2].

Under circumstances above the following values are the same:

- (i) $\sup\{\phi^M : \text{for all } M \models \Gamma = 0\}$;
- (ii) $\inf\{p \in \mathbb{Q} : \Gamma \vdash \phi \dot{-} p\}$.

We denote this value by ϕ° .

If the language L is computable, the set of all continuous L -formulas and the set of all L -conditions of the form

$$\phi \leq \frac{m}{n}, \text{ where } \frac{m}{n} \in \mathbb{Q}_+,$$

are computable. Moreover if Γ is a computably enumerable set of formulas, then the relation $\Gamma \vdash \phi$ is computably enumerable.

Corollary 9.11 of [2] states that when Γ is computably enumerable and $\Gamma = 0$ axiomatises a complete theory, then the value of ϕ with respect to Γ is a recursive real which is uniformly computable from ϕ . This exactly means that the corresponding complete theory is *decidable* (see Section 3). Note that in this case the value of ϕ coincides with ϕ° .

2 Dynamical n -qubit spaces

We treat a Hilbert space over \mathbb{R} exactly as in Section 15 of [1]. We identify it with a many-sorted metric structure

$$(\{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_r\}_{r \in \mathbb{R}}, +, -, \langle \rangle),$$

where B_n is the ball of elements of norm $\leq n$, $I_{mn} : B_m \rightarrow B_n$ is the inclusion map, $\lambda_r : B_m \rightarrow B_{km}$ is scalar multiplication by r , with k the unique integer satisfying $k \geq 1$ and $k - 1 \leq |r| < k$; furthermore, $+, - : B_n \times B_n \rightarrow B_{2n}$ are vector addition and subtraction and $\langle \rangle : B_n \rightarrow [-n^2, n^2]$ is the predicate of the inner product. The metric on each sort is given by $d(x, y) = \sqrt{\langle x - y, x - y \rangle}$. For every operation the continuity modulus is standard. For example in the case of λ_r this is $\frac{z}{|r|}$.

Stating existence of infinite approximations of orthonormal bases by axioms of the form

$$\inf_{x_1, \dots, x_n} \max_{1 \leq i < j \leq n} (|\langle x_i, x_j \rangle - \delta_{i,j}|) = 0, \quad n \in \omega,$$

$$\delta_{i,j} \in \{0, 1\} \text{ with } \delta_{i,j} = 1 \leftrightarrow i = j,$$

we axiomatise infinite dimensional Hilbert spaces. By [1] they form the class of models of a complete theory which is κ -categorical for all infinite κ , and admits elimination of quantifiers.

When we assume that the space is finite dimensional all sorts B_n become compact. This case corresponds to the case of finite structures in ordinary model theory. The dimension can be described by maximal n so that the following sentence holds.

$$\inf_{y_1, \dots, y_n} \sup_x (|\langle x, x \rangle|^2 - |\langle x, y_1 \rangle|^2 - \dots - |\langle x, y_n \rangle|^2) = 0.$$

The corresponding continuous theory admits elimination of quantifiers. This follows by the argument of Lemma 15.1 from [1].

This approach can be naturally extended to complex Hilbert spaces,

$$(\{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_c\}_{c \in \mathbb{C}}, +, -, \langle \rangle_{Re}, \langle \rangle_{Im}).$$

We only extend the family $\lambda_r : B_m \rightarrow B_{km}$, $r \in \mathbb{R}$, to a family $\lambda_c : B_m \rightarrow B_{km}$, $c \in \mathbb{C}$, of scalar products by $c \in \mathbb{C}$, with k the unique integer satisfying $k \geq 1$ and $k - 1 \leq |c| < k$.

We also introduce *Re*- and *Im*-parts of the inner product.

If we remove from the signature of complex Hilbert spaces all scalar products by $c \in \mathbb{C} \setminus \mathbb{Q}[i]$, we obtain a countable subsignature

$$(\{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_c\}_{c \in \mathbb{Q}[i]}, +, -, \langle \rangle_{Re}, \langle \rangle_{Im}),$$

which is *dense* in the original one:

if we present $c \in \mathbb{C}$ by a sequence $\{q_i\}$ from $\mathbb{Q}[i]$ converging to c , then the choice of the continuity moduli of the restricted signature still guarantees that in any sort B_n the functions λ_{q_i} form a sequence which converges to λ_c with respect to the metric

$$\sup_{x \in B_n} \{|f^M(x) - g^M(x)| : M \text{ is an } L\text{-structure}\}.$$

This obviously implies that the original language of Hilbert spaces is *separable*. In particular we may apply Remark 1.1.

We extend structures of complex Hilbert spaces by additional discrete sort Q with $\{0, 1\}$ -metric and a map $qu : Q \rightarrow B_1$ so that the set $qu(Q)$ is an orthonormal basis of \mathbb{H} .

When Q consists of 2^n elements we may denote them by $|i_0 \dots i_{n-1}\rangle$ with $i_j \in \{0, 1\}$. In fact this is the **n -qubit space**. It is distinguished by the axiom

$$\sup_x (|\langle x, x \rangle|^2 - |\langle x, qu(q_0) \rangle|^2 - \dots - |\langle x, qu(q_i) \rangle|^2 - \dots - |\langle x, qu(q_{2^n-1}) \rangle|^2) = 0$$

To study dynamical evolutions of quantum circuits we introduce the following expansion of n -qubit spaces. Let us fix a natural number t and consider the class of **dynamical n -qubit spaces** (i.e. 2^n -dimensional) in the extended signature

$$(Q, qu, \{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_c\}_{c \in \mathbb{Q}[i]}, +, -, \langle \rangle_{Re}, \langle \rangle_{Im}, U_1, \dots, U_t),$$

where U_j , $1 \leq j \leq t$, are symbols of unitary operators of \mathbb{H} . We may assume that all U_j are defined only on B_1 .

Lemma 2.1 *The complete continuous theory of each structure of this form is axiomatised by the standard axioms of n -qubit spaces, the axioms stating that each U_j is a unitary operator and the axioms describing the matrix of U_j in the basis $qu(Q)$ (appropriately enumerated by $1, \dots, N = 2^n$):*

$$\inf_{q_1 \dots q_N} \sup_{j,l} (\| U_j(qu(q_l)) - \sum \lambda_{c_k}(qu(q_k)) \|) \leq \varepsilon_l, \text{ where } \varepsilon_l \in \mathbb{Q}, c_k \in \mathbb{Q}[i].$$

Indeed in any model with these axioms the values of $U_j(qu(q_l))$ have the same coordinates in appropriately enumerated $qu(Q)$. Thus the lemma is obvious.

Each dynamical n -qubit space defines a family of quantum automata over the language $\{1, \dots, t\}^*$, where each automaton is determined by the 2^n -dimensional diagonal matrix P of the projection to final states of $qu(Q)$.

Fixing $\lambda \in \mathbb{Q}$ we say that a word $w = i_1 \dots i_k$ is *accepted* by the corresponding P -automaton if

$$ACC_w = \| PU_{i_k} \dots U_{i_1} |0^{\otimes n}\rangle \|^2 > \lambda.$$

The corresponding algorithmic problems were in particular studied in the paper of H.Derksen, E.Jeandel, P.Koiran [5]. They have proved that the following problems are decidable for U_1, \dots, U_t over finite extensions of $\mathbb{Q}[i]$:

- (i) Is there w such that $ACC_w > \lambda$?
- (ii) Is a threshold λ isolated, i.e. is there ε that for all w , $|ACC_w - \lambda| \geq \varepsilon$?
- (iii) Is there a threshold λ which is isolated?

The observation that given P each statement $ACC_w \leq \lambda$ or $|ACC_w - \lambda| \geq \varepsilon$ can be rewritten as a continuous statement of the theory of dynamical n -qubit spaces partially motivated our research in this paper.

Describe classes of qubit spaces (possibly with additional operators) having decidable continuous theory.

Below we study continuous statements θ so that the continuous theory of (dynamical) qubit spaces satisfying θ is decidable.

3 Decidability/undecidability of continuous theories

In this section we assume that the signature L is computable and values of formulas are in $[0, 1]$. The interval $[0, 1]$ can be obviously replaced by any compact interval. We start with the following definition from [2].

Definition 3.1 *A continuous theory T is called **decidable** if for every sentence ϕ the degree of truth*

$$\phi^\circ = \sup\{\phi^M : M \models T\}$$

is a computable real which is uniformly computable from ϕ .

This exactly means that there is an algorithm which for every ϕ and a rational number δ finds a rational r such that $|r - \phi^\circ| \leq \delta$.

Note that decidability of T does not imply that the set of all continuous ϕ with $\phi^M = 0$ for all $M \models T$, is computable. On the other hand it is easy to see that decidability of T follows from this condition.

The following theorem is a counterpart of Ershov's decidability criterion (Theorem 6.1.1 of [9]). Here we call a sequence of complete continuous theories $\{T_i, i \in \omega\}$ **effective** if the relation

$$\{(\theta, j) : \theta \text{ is a statement so that } T_j \vdash \theta\}$$

is computably enumerable.

Theorem 3.2 *A continuous theory T is decidable if and only if T can be defined by a computably enumerable system of axioms and T can be presented $T = \bigcap_{i \in \omega} T_i$ where $\{T_i, i \in \omega\}$ is an effective sequence of complete continuous theories.*

Proof. Sufficiency. Let ϕ be a continuous sentence. For every natural n we can apply an effective procedure which looks for conditions of the form $\phi \leq \frac{k}{n}$ derived from the axioms of T and conditions of the form $\frac{l}{n} \leq \phi$ which appear in some $T_j \vdash \frac{l}{n} \leq \phi$. This always gives a number $k < n - 1$ such that $\frac{k}{n} \leq \phi \leq \frac{k+2}{n}$.

Necessity. For every sentence ϕ we fix a computably enumerable sequence of segments $[l_{n,\phi}, r_{n,\phi}]$ converging to ϕ° so that $\phi^\circ \in [l_{n,\phi}, r_{n,\phi}]$. Then all statements $\phi \leq r_{n,\phi}$ form a computably enumerable sequence of axioms of T .

Now for every sentence ϕ we effectively build a complete theory $T_{n,\phi} \supset T$ with $T_{n,\phi} \vdash l_{n,\phi} \dot{-} \phi \leq 2^{-n}$. In fact such a construction produces an effective family $T_i, i \in \omega$, from the formulation. Indeed, then for every natural n we can find a sufficiently large m so that $T_{m,\phi} \vdash \phi^\circ \dot{-} \phi \leq 2^{-n}$ (here ϕ° is defined by T). This obviously implies that T coincides with the intersection of all $T_{m,\phi}$. Effectiveness will be verified below.

At Step 0 we define $T_{n,\phi,0}$ to be the extension of T by the axiom $l_{n,\phi} \dot{-} \phi \leq 0$. At every step $m + 1$ we build a finite extension $T_{n,\phi,m+1}$ of T so that each inequality $\psi \leq 0$ from $T_{n,\phi,m} \setminus T$ is transformed into an inequality $\psi \leq \varepsilon$, where $\varepsilon \leq 2^{-(2n+m+1)}$. The 'limit theory' $T_{n,\phi} = \lim_{m \rightarrow \infty} T_{n,\phi,m}$ is defined by the limits of these values ε for all formulas ψ . Note that it can happen that $\varepsilon \leq 0$, i.e. the transformed inequality is of the form $\psi + \delta \leq 0$, with $\delta > 0$. On the other hand we will see that for every ψ the axioms of $\lim_{m \rightarrow \infty} T_{n,\phi,m}$ give an effective sequence of rational numbers which converges to the value of ψ under this theory.

Let us enumerate all triples (n, ϕ, ψ) by natural numbers > 0 so that each triple has infinitely many numbers. Assume that the number $m + 1$ codes a triple (n, ϕ, ψ) . For all $n' \neq n$ we put $T_{n',\phi',m+1} = T_{n',\phi',m}$. Assume that at Step m the theory $T_{n,\phi,m} \setminus T$ already contains inequalities $\frac{k_l}{l} \leq \psi_l \leq \frac{k'_l}{l}$ for some natural l and $k_l, k'_l \leq l$. In particular we admit that the 0-th inequality $l_{n,\phi} \dot{-} \phi \leq 0$ has been already transformed into an inequality $l_{n,\phi} \dot{-} \phi \leq \varepsilon$ for some $\varepsilon \leq \sum_{i \leq m} 2^{-(2n+i+1)}$. Let θ be

$$\psi \dot{-} 2^{2n+m+2} \max_l (\max(\psi_l \dot{-} \frac{k'_l}{l}, \frac{k_l}{l} \dot{-} \psi_l, l_{n,\phi} \dot{-} (\phi + \sum_{i \leq m} 2^{-(2n+i+1)}))).$$

Since T is decidable we compute $k_{m+1} < m$ so that $\frac{k_{m+1}}{m+1} \leq \theta^\circ \leq \frac{k_{m+1}+2}{m+1}$. Then the value of ψ under $T_{n,\phi,m}$ is not greater than $\frac{k_{m+1}+2}{m+1}$. This means that extending $T_{n,\phi,m}$ by $\psi \leq \frac{k_{m+1}+2}{m+1}$ we preserve consistency of the theory. If $k_{m+1} = 0$ this finishes our construction at this step.

If $k_{m+1} > 0$ we need an additional correction. Let θ' be

$$\psi \dot{-} 2^{2n+m+2} \max_l \left(\max \left(\psi_l \dot{-} \frac{k'_l}{l}, \frac{k_l}{l} \dot{-} \psi_l, l_{n,\phi} \dot{-} \left(\phi + \sum_{i \leq m} 2^{-(2n+i+1)} \right), \psi \dot{-} \frac{k_{m+1}+2}{m+1} \right) \right).$$

Since T is decidable we compute $k'_{m+1} < m$ so that $\frac{k'_{m+1}}{m+1} \leq (\theta')^\circ \leq \frac{k'_{m+1}+2}{m+1}$. Then the value of ψ under $T_{n,\phi,m}$ together with $\psi \leq \frac{k_{m+1}+2}{m+1}$ is not greater than $\frac{k'_{m+1}+2}{m+1}$. This means that extending $T_{n,\phi,m}$ by $\psi \leq \frac{\min(k_{m+1}, k'_{m+1})+2}{m+1}$ we preserve consistency of the theory.

If $0 < k'_{m+1} < k_{m+1}$ we repeat this construction again. It is clear that finally we arrive at the situation when after such a repetition the number k_{m+1} does not change. Then note that the value ψ° under the extension of T by $\psi \leq \frac{k_{m+1}+2}{m+1} + 2^{-(2n+m+2)}$ and all statements of the form

$$\frac{k_l}{l} - 2^{-(2n+m+2)} \leq \psi_l \leq \frac{k'_l}{l} + 2^{-(2n+m+2)} \quad (\text{for inequalities } \frac{k_l}{l} \leq \psi_l \leq \frac{k'_l}{l} \text{ from } T_{n,\phi,m})$$

satisfies $\frac{k_{m+1}}{m+1} \leq \psi^\circ$ (apply $\frac{k_{m+1}}{m+1} \leq \theta^\circ$ with respect to T). We now define $T_{n,\phi,m+1}$ as the set of so corrected statements of $T_{n,\phi,m}$ together with the statement

$$\frac{k_{m+1}}{m+1} \leq \psi \leq \frac{k_{m+1}+2}{m+1} + 2^{-(2n+m+2)}.$$

It is clear that the obtained extension is consistent with T .

By the choice of a repeating enumeration we see that for each sentence ψ boundaries of ψ at steps of our procedure form a Cauchy sequence. Thus ψ has the same value in all models of $T_{n,\phi}$. Moreover the inequality $l_{n,\phi} \dot{-} \phi \leq 0$ will be transformed into $l_{n,\phi} \dot{-} \phi \leq 2^{-n}$. We see that Step 0 guarantees that T coincides with the intersection of all $T_{n,\phi}$.

Note that after the $(m+1)$ -th step we know that for every inequality $\psi' \leq \delta$ from each $T_{n',\phi',m+1} \setminus T$ the upper boundary of ψ' in the final $T_{n',\phi'}$ cannot exceed $\delta + \frac{1}{2^m}$. In particular all inequalities of this kind can be included into an enumeration of axioms of $T_{n',\phi'}$ at this step. Thus we see that by the effectiveness of our procedure the family $\{T_{n,\phi}\}$ is effective. \square

In order to have a method for proving undecidability of continuous theories we now discuss interpretability of first order structures in continuous ones.

Let $L_0 = \langle P_1, \dots, P_m \rangle$ be a finite relational signature. Let \mathcal{K}_0 be a class of finite first-order L_0 -structures. Let \mathcal{K} be a class of continuous L -structures, where L is as above. We say that \mathcal{K}_0 is **relatively interpretable** in \mathcal{K} if there is a finite constant

extension $L(\bar{a}) = L \cup \{a_1, \dots, a_r\}$, a constant expansion $\mathcal{K}(\bar{a})$ of \mathcal{K} and there are continuous L -formulas

$$\phi^-(\bar{x}, \bar{y}), \phi^+(\bar{x}, \bar{y}), \theta^-(\bar{x}, \bar{y}_1, \bar{y}_2), \theta^+(\bar{x}, \bar{y}_1, \bar{y}_2) \text{ and}$$

$$\psi_1^-(\bar{x}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_1}), \psi_1^+(\bar{x}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_1}), \dots, \psi_m^-(\bar{x}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_m}), \psi_m^+(\bar{x}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_m}),$$

with $|\bar{y}| = |\bar{y}_1| = |\bar{y}_2| = \dots = |\bar{y}_{l_j}| = \dots = |\bar{y}_{l_m}|$, such that:

- (i) the L -reduct of $\mathcal{K}(\bar{a})$ coincides with \mathcal{K} ;
- (ii) the conditions $\phi^-(\bar{a}, \bar{y}) \leq 0$ and $\phi^+(\bar{a}, \bar{y}) > 0$ are equivalent in any $M \in \mathcal{K}(\bar{a})$ and the condition $\theta^-(\bar{a}, \bar{y}_1, \bar{y}_2) \leq 0$ defines an equivalence relation on the zero-set of $\phi^-(\bar{a}, \bar{y})$ (on tuples of the corresponding power M^s with $s = |\bar{y}_1|$), so that the values of any $\psi_i^\varepsilon(\bar{a}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_i})$ are invariant under this equivalence relation;
- (iii) the (+)-conditions below are equivalent to (−)-ones in $\mathcal{K}(\bar{a})$:

$$\theta^-(\bar{a}, \bar{y}_1, \bar{y}_2) \leq 0, \theta^+(\bar{a}, \bar{y}_1, \bar{y}_2) > 0, \psi_1^-(\bar{a}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_1}) \leq 0,$$

$$\psi_1^+(\bar{a}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_1}) > 0, \dots, \psi_m^-(\bar{a}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_m}) \leq 0, \psi_m^+(\bar{a}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_m}) > 0;$$

- (iv) for any $M \in \mathcal{K}(\bar{a})$ the conditions of (iii) define an L_0 -structure from \mathcal{K}_0 on the θ -quotient of the zero-set of $\phi^-(\bar{a}, \bar{y})$ and any structure of \mathcal{K}_0 can be so realised.

Theorem 3.3 *Under circumstances above assume that $Th(\mathcal{K}_0)$ is undecidable. Then the continuous theory $Th(\mathcal{K}(\bar{a}))$ is not a computable set.*

Proof. The proof is straightforward. To each formula ψ of the theory of \mathcal{K}_0 so that the quantifier-free part is in the disjunctive normal form we associate the appropriately rewritten continuous formula $\psi^-(\bar{a})$ (with appropriate free variables). In particular atomic formulas are written by (−)-conditions above, but negations of atomic formulas appear in the form of

$$\psi_i^+(\bar{a}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{l_i}) \leq 0.$$

Condition (ii) and the condition that the θ -quotient of the zero-set of $\phi^-(\bar{a}, \bar{y})$ is always finite, allow us to use standard quantifiers in such $\psi^-(\bar{a})$: the quantifier \forall is written as *sup* but \exists is written as *inf*. Note that if ψ' is equivalent to $\neg\psi$ then $(\psi')^-(\bar{a}) \leq 0$ is equivalent to $\psi^-(\bar{a}) > 0$ for tuples from the zero-set of $\phi^-(\bar{a}, \bar{y})$.

This construction reduces the decision problem for $Th(\mathcal{K}_0)$ to computability of $Th(\mathcal{K}(\bar{a}))$. \square

It is worth noting that in the classical first-order logic the situation of this theorem usually has much stronger consequences. For example Theorem 5.1.2 of [9] in a slightly modified setting (and removing the assumption that \mathcal{K}_0 consists of finite structures) states that hereditary undecidability of $Th(\mathcal{K}_0)$ can be lifted to $Th(\mathcal{K})$. The 'positiveness' of the continuous logic does not allow so strong statements.

As we already know the statement of Theorem 3.3 does not imply that $Th(\mathcal{K}(\bar{a}))$ is undecidable. To prove the undecidability theorems from Section 4 we will use an additional tool. It will be applied in a combination with the method of Theorem 3.3.

4 Decidability/undecidability of theories of qubit spaces

We start this section with an undecidability result of some classes of constant expansions of qubit spaces. Then we apply the idea of the proof to a more interesting example of a class of dynamical qubit spaces.

Theorem 4.1 *There is a class of qubit spaces expanded by four constants, i.e. structures of the form*

$$(Q, qu, \{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_c\}_{c \in \mathbb{Q}[i]}, +, -, \langle \rangle_{Re}, \langle \rangle_{Im}, a_1, a_2, b_1, b_2)$$

which is distinguished in the class of all qubit spaces with $\langle b_1, b_2 \rangle \neq 0$ by a continuous statement and which has undecidable continuous theory.

Proof. Consider the following formula:

$$\psi(x, y_1, y_2) = |\langle qu(y_1), x \rangle - \langle qu(y_2), x \rangle|,$$

where y_1, y_2 are variables of the sort Q and x is of the sort B_1 .

In any qubit space any equivalence relation on Q can be realised by $\psi(a, y_1, y_2) \leq 0$ for appropriately chosen $a \in B_1$: define a to be a linear combination of $qu(q_l)$, so that for equivalent q_j and q_k the coefficients of $qu(q_j)$ and $qu(q_k)$ in a are the same.

Let us introduce the following formula:

$$\psi^c(x, z_1, z_2, y_1, y_2) = |\langle z_1, z_2 \rangle| \dot{-} |\langle qu(y_1), x \rangle - \langle qu(y_2), x \rangle|,$$

where y_1, y_2 are variables of the sort Q and x, z_1, z_2 are of B_1 .

If a defines an equivalence relation on Q as above then there are $b_1, b_2 \in B_1$ with $\langle b_1, b_2 \rangle \neq 0$ and

$$\sup_{y_1, y_2} \min(\psi(a, y_1, y_2), \psi^c(a, b_1, b_2, y_1, y_2)) \leq 0$$

$$\sup_{y_1, y_2} (|\langle b_1, b_2 \rangle| \dot{-} (\psi(a, y_1, y_2) + \psi^c(a, b_1, b_2, y_1, y_2))) \leq 0.$$

We see that the formula $\psi^c(a, b_1, b_2, y_1, y_2)$ can be interpreted as the complement of the equivalence relation defined by $\psi(a, y_1, y_2)$ in the class of these qubit spaces.

This allows us to define interpretability of the first-order theory of finite structures of two equivalence relations (which is undecidable by Proposition 5.1.7 from [9]) in the class, say \mathcal{K} , of qubit spaces extended by constants a_1, a_2, b_1, b_2 where b_1, b_2 satisfy the statements above for both a_1 and a_2 instead of a .

In fact we axiomatise \mathcal{K} in the class of all qubit spaces with $\langle b_1, b_2 \rangle \neq 0$ by the following continuous statements:

$$\sup_{y_1, y_2} \min(\psi(a_i, y_1, y_2), |\langle b_1, b_2 \rangle| \dot{-} |\langle qu(y_1), a_i \rangle - \langle qu(y_2), a_i \rangle|) \leq 0, \text{ where } i = 1, 2.$$

In terms of Theorem 3.3 the formulas $\phi^+, \phi^-, \theta^+, \theta^-$ become degenerate: ϕ^- can be taken as $d(y, y)$ for the (descrete) sort Q , then

$$\phi^+(y) = 1 \dot{-} d(y, y), \theta^-(y_1, y_2) = d(y_1, y_2), \theta^+(y_1, y_2) = 1 - d(y_1, y_2).$$

Formulas $\psi(a_1, y_1, y_2)$ and $\psi^c(a_1, b_1, b_2, y_1, y_2)$ play the role of ψ_1^- and ψ_1^+ . Then $\psi(a_2, y_1, y_2)$ and $\psi^c(a_2, b_1, b_2, y_1, y_2)$ play the role of ψ_2^- and ψ_2^+ .

To each formula $\rho(\bar{y})$ of the theory of two equivalence relations so that the quantifier-free part is in the disjunctive normal form we associate the appropriately rewritten continuous formula $\rho^*(a_1, a_2, b_1, b_2, \bar{y})$ (where we use \min and \max instead of \vee and \wedge). Since the free variables of the latter \bar{y} are of the sort Q , when ρ is quantifier-free, the values $\rho^*(a_1, a_2, b_1, b_2, \bar{c})$ belong to $\{0\} \cup [|\langle b_1, b_2 \rangle|, 1]$. It is easy to see that in structures of \mathcal{K} the same property holds for any formula $\rho(\bar{y})$.

This obviously implies that when ρ is a sentence, the sentence

$$\min(|\langle b_1, b_2 \rangle|, \rho^*(a_1, a_2, b_1, b_2))$$

has the following property:

ρ is satisfied in all finite models of two equivalence relations if and only if all structures of \mathcal{K} satisfy $\min(|\langle b_1, b_2 \rangle|, \rho^*(\bar{a}, \bar{b})) = 0$.

If the theory of \mathcal{K} was decidable, it would define the real number $|\langle b_1, b_2 \rangle|^o$ which would be computable and $\neq 0$. In particular we could compute a rational $r > 0$, so that $\min(|\langle b_1, b_2 \rangle|, \rho^*(\bar{a}, \bar{b})) = 0$ is equivalent to $\min(|\langle b_1, b_2 \rangle|, \rho^*(\bar{a}, \bar{b})) \leq r$. By Theorem 3.3 this is impossible. \square

Theorem 4.2 *There is a class of dynamical qubit spaces in the signature*

$$(Q, qu, \{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_c\}_{c \in \mathbb{Q}[i]}, +, -, \langle \rangle_{Re}, \langle \rangle_{Im}, U_1, U_2, U_3, U_4, U_5)$$

which is distinguished in the class of all qubit spaces with

$$\sup_v d(U_3(v), v) \neq 0$$

by a continuous statement and which has undecidable continuous theory.

Proof. We will use the construction of Theorem 4.1 with some necessary changes. For example we replace the value $|\langle b_1, b_2 \rangle|$ from that theorem by $\sup_v d(U_3(v), v)$. The constants a_1 and a_2 will appear as the normed vectors fixed by U_1 and U_2 respectively. Although we choose U_i , $i = 1, 2$, so that the subspace of fixed vectors of U_i coincides with $\mathbb{C}a_i$, we cannot define these constants by a continuous formula. This is why some additional values will be used in the proof. The values $\sup_v d(U_3(v), v)$ and $\sup_v d(U_4(v), v)$ will appear in 'fuzzy' versions of formulas from the proof of Theorem 4.1.

Let:

$$\begin{aligned} \psi_i(y_1, y_2) = & \sup_u \min(\sup_{v_1} (d(U_3(v_1), v_1)) \dot{-} \max(d(U_i(u), u), |1 - \|u\| |), \\ & (|\langle qu(y_1), u \rangle - \langle qu(y_2), u \rangle| \dot{-} \sup_{v_2} d(U_4(v_2), v_2))), \end{aligned}$$

where y_1, y_2 are variables of the sort Q , $i \in \{1, 2\}$ and u, v_1, v_2 are of the sort B_1 .

To see that in any qubit space any equivalence relation on Q can be realised by $\psi_i(y_1, y_2) \leq 0$ let us define a_1 (the case of a_2 is similar) to be a linear combination of $qu(q_l)$ of length 1, so that for equivalent q_j and q_k the coefficients of $qu(q_j)$ and $qu(q_k)$ in a_1 are the same. We also fix a rational number r (for both a_1 and a_2) so that for non-equivalent q_j and q_k the coefficients of $qu(q_j)$ and $qu(q_k)$ in a_1 are distant by $> r$. Note that any $e^{i\phi}a_1$ has the same properties with respect to elements of $qu(Q)$.

We extend a_1 to an orthonormal basis of \mathbb{H} and define U_1 to be a unitary operator having these vectors as eigenvectors so that $\mathbb{C}a_1$ is the subspace of fixed points. The remaining eigenvalues are chosen so that the corresponding eigenvectors are taken by U_i at the distance $\geq 1/10$.

Now we can take the operators U_3 and U_4 so close to Id (with respect to the operator norm) that the formula ψ_1 indeed realises the equivalence relation we consider. Choosing U_4 we demand that $\sup_{v_2}(d(U_4(v_2), v_2))$ is much less than r . Having U_4 we can find U_3 . We will assume that $\sup_v d(U_3(v), v) > 0$ in our structures.

Let us now introduce U_5 with $r = \sup_v(d(U_5(v), v))$ and consider the following formulas for $i = 1, 2$:

$$\begin{aligned} \psi_i^c(y_1, y_2) = & \sup_u \min(\sup_{v_1}(d(U_3(v_1), v_1)) \dot{-} \max(d(U_i(u), u), |1 - \|u\| |)), \\ & (\sup_{v_2} d(U_5(v_2), v_2) \dot{-} |\langle qu(y_1), u \rangle - \langle qu(y_2), u \rangle|)), \end{aligned}$$

where y_1, y_2 are variables of the sort Q and u, v_1, v_2 are of B_1 . If necessary we may correct U_3 making $\sup_v d(U_3(v), v)$ smaller so that the following statements hold.

$$\begin{aligned} \sup_{y_1, y_2} \min(\psi_i(y_1, y_2), \psi_i^c(y_1, y_2)) & \leq 0, \\ \sup_{y_1, y_2} (\sup_v d(U_3(v), v) \dot{-} (\psi_i(y_1, y_2) + \psi_i^c(y_1, y_2))) & \leq 0. \end{aligned}$$

As before the formula $\psi_i^c(y_1, y_2)$ will be interpreted as the complement of the equivalence relation defined by $\psi_i(y_1, y_2)$ in the class of these qubit spaces.

This allows us to define interpretability of the (undecidable) first-order theory of finite structures of two equivalence relations in the class, say \mathcal{K} , of dynamical qubit spaces with respect to operators U_1, U_2, U_3, U_4, U_5 . The formulas $\phi^+, \phi^-, \theta^+, \theta^-$ (see Theorem 3.3) are taken as in Theorem 4.1 (i.e. $\phi^-(y) = d(y, y)$ and $\theta^-(y_1, y_2) = d(y_1, y_2)$). Formulas $\psi_i(y_1, y_2)$ and $\psi_i^c(y_1, y_2)$ play the role of ψ_i^- and ψ_i^+ for $i = 1, 2$.

To each formula $\rho(\bar{y})$ of the theory of two equivalence relations so that the quantifier-free part is in the disjunctive normal form we associate the appropriately rewritten continuous formula $\rho^*(\bar{y})$. Since the free variables of the latter \bar{y} are of the sort Q , when ρ is quantifier-free, the values $\rho^*(\bar{c})$ belong to $\{0\} \cup [\sup_v d(U_3(v), v), 1]$. Thus we see that in structures of \mathcal{K} the same property holds for any formula $\rho(\bar{y})$.

This obviously implies that when ρ is a sentence, the sentence

$$\min(\sup_v d(U_3(v), v), \rho^*)$$

has the following property:

ρ is satisfied in all finite models of two equivalence relations if and only if all structures of \mathcal{K} satisfy $\min(\sup_v d(U_3(v), v), \rho^*) = 0$.

If the theory of \mathcal{K} was decidable, it would define the real number $\sup_v d(U_3(v), v)^o$ which would be computable and $\neq 0$. In particular we could compute a rational $r' > 0$, so that $\min(\sup_v d(U_3(v), v), \rho^*) = 0$ is equivalent to $\min(\sup_v d(U_3(v), v), \rho^*) \leq r'$. By Theorem 3.3 this is impossible. \square

Let us now restrict the dimension of qubit spaces, say by 2^n . It is natural to expect that then the theory of (dynamical) qubit spaces becomes decidable. In classical model theory this corresponds to the situation of a theory of structures of a fixed finite size.

On the other hand since the structures are of infinite language it is not very difficult to find such a structure with undecidable continuous theory. For example one can take a dynamical qubit space with one additional operator which takes a basic vector, say $qu(q_0)$, to some linear combination $r \cdot qu(q_0) + \sqrt{1-r^2} \cdot qu(q_1)$ where r is a non-computable real number.

Let us fix a signature

$$(Q, qu, \{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_c\}_{c \in \mathbb{Q}[i]}, +, -, \langle \rangle_{Re}, \langle \rangle_{Im}, U_1, \dots, U_t),$$

where as before we assume that U_j , $1 \leq j \leq t$, are symbols of unitary operators of \mathbb{H} which are defined only on B_1 . Using Theorem 3.2 we will prove that the theory of 2^n -qubit spaces in this language is decidable.

Let us enumerate all 2^n -dimensional unitary matrices of computable complex numbers (i.e. their real and imaginary parts are defined by computable sequences). This induces an enumeration Axm_j , $j \in \omega$, of systems of axioms of complete continuous theories T_j of dynamical n -qubit spaces. Each Axm_j consists of the standard axioms of n -qubit spaces, the axioms stating that each U_s is a unitary operator and the axioms describing the matrix of U_s in the basis $qu(Q)$:

$$\inf_{q_1 \dots q_N} \|U_s(qu(q_l)) - \sum \lambda_{c_j}(qu(q_j))\| \leq \varepsilon_l, \text{ where } \varepsilon \in \mathbb{Q}, c_j \in \mathbb{Q}[i].$$

Using Lemma 2.1 it is easy to see that the enumeration Axm_j , $j \in \omega$, gives an effective indexation of complete continuous theories T_i of dynamical n -qubit spaces in the sense of Section 3. The statement that the relation $\{(\theta, j) : \theta \text{ is a statement so that } T_j \vdash \theta\}$ is computably enumerable follows from the fact that this relation coincides with $\{(\theta, j) : \theta \text{ is a statement so that } Axm_j \vdash \theta\}$.

Theorem 4.3 *The theory of all dynamical n -qubit spaces coincides with the intersection $\bigcap T_j$.*

The theory of all dynamical n -qubit spaces is decidable.

Proof. By Theorem 3.2 the second statement of the theorem follows from the first one. Thus we only have to show that for any rational δ , any dynamical n -qubit space

$$(Q, qu, \{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_c\}_{c \in \mathbb{Q}[i]}, +, -, \langle \rangle_{Re}, \langle \rangle_{Im}, U_1, \dots, U_t),$$

and any continuous sentence $\theta(U_1, \dots, U_t)$ over this structure there are operators $\tilde{U}_1, \dots, \tilde{U}_t$ defined by matrices over $\mathbb{Q}[i]$, so that

$$|\theta(U_1, \dots, U_t) - \theta(\tilde{U}_1, \dots, \tilde{U}_t)| \leq \delta.$$

Indeed this shows that when some $\theta(U_1, \dots, U_t) \leq \varepsilon$ does not belong to T , then it does not belong to some T_j .

Since any continuous formula defines a uniformly continuous function and the ball B_1 is compact it suffices to take $\tilde{U}_1, \dots, \tilde{U}_t$ so that they sufficiently approximate U_1, \dots, U_t . This is a folklore fact. On the other hand it is a curious place where the following fact from quantum computations can be applied.

Let $CNOT$ be a 2-qubit linear operator defined by

$$CNOT : |00\rangle \rightarrow |00\rangle, |01\rangle \rightarrow |01\rangle, |10\rangle \rightarrow |11\rangle, |11\rangle \rightarrow |10\rangle.$$

The *Toffoli gate* is a 3-qubit linear operator defined on basic vectors by

$$\Lambda(CNOT) : |\varepsilon_1 \varepsilon_2 \varepsilon_3\rangle \rightarrow |\varepsilon_1 \varepsilon_2 (\varepsilon_3 \oplus \varepsilon_1 \cdot \varepsilon_2)\rangle.$$

Let \mathcal{B} be a 2-dimensional space over \mathbb{C} . It is well-known that

- (a) Any unitary transformation of $(\mathcal{B})^{\otimes k}$ is a product of 1-qubit unitary transformations and 2-qubit copies of $CNOT$ at appropriate registers.
- (b) The operators of the basis

$$\mathcal{Q} = \left\{ K = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, CNOT, \Lambda(CNOT), \text{Hadamard's } H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right\}$$

generate a dense subgroup of $U(\mathcal{B}^{\otimes 3})/U(1)$ under the operator norm. \square

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